J. Fluid Mech. (1984), *vol.* 149, *pp.* 431–443
Printed in Great Britain

A seventeenth-order series expansion for the solitary wave

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(Received 15 November 1982 and in revised form 5 May 1984)

The properties of solitary waves are investigated numerically using a series in $\operatorname{sech}^2 \frac{1}{2}x$ to describe the wave profile. (It is shown that this expansion is complete in the L^2 sense.) Seventeen terms in the series are computed. For waves of amplitudes up to half the undisturbed fluid depth, the 17-term partial sum gives profiles and wave-parameter values with at least two-digit accuracy. For waves of larger amplitude, if Padé approximants are used to accelerate convergence, the computed values of the wave parameters are found to agree with the values obtained by Longuet-Higgins & Fenton (1974), but differ from those of Williams (1981), Witting (1981) and Hunter & Vanden-Broeck (1983). Possible explanations of this discrepancy are discussed.

1. Introduction

The solitary wave has been the subject of a great deal of study since the report of its discovery by John Scott Russell (Russell 1838). Nevertheless, an exact description for a wave of arbitrary height has yet to be found, and the maximum possible height for a solitary wave in a channel of constant depth is still not known exactly. A list of several approximate values for the maximum amplitude is presented in table 1. All of these estimates (except Laitone's) rely on Stokes' criterion for the determination of the wave of maximum amplitude. Stokes (1880) conjectured that the highest-possible stationary wave would be characterized by sharp crests, the fluid velocity at the crests then necessarily vanishing (in the frame of reference in which the flow is steady). He went on to demonstrate that the crests would have to enclose an angle of 120°. This conjecture has apparently been proved recently for the solitary wave (see the remark added to the paper of Amick & Toland 1981).

Longuet-Higgins & Fenton (1974) computed the values of several parameters for solitary waves of all amplitudes (up to the maximum) by means of partial power series and Padé approximants. Byatt-Smith & Longuet-Higgins (1976) also computed parameter values for waves of almost all amplitudes using Byatt-Smith's (1970) integral-equation formulation. In addition, they were able to plot profiles for all waves except those of nearly maximum amplitude. Williams (1981) studied periodic waves of maximum height by explicitly including two terms designed to produce sharp crests of the proper form. He provides extensive tabulations of profiles and pressure, velocity and acceleration distributions. His large-wavelength data may be applied to solitary waves. Witting's (1981) study of solitary waves was based on a Fourier-series technique, with a singular term to give the sharp crest of the highest wave. More

Author	maximum amplitude / h_0
Russell (experimental, 1845)	1
Boussinesq (see Keulegan & Patterson 1940)	0.73
Rayleigh (1876)	1
McCowan (1891)	0.82
McCowan (1894)	0.78
Laitone (1960)	$\frac{8}{11}$
Lenau (1966)	0.83
Yamada, Limara & Okabe (1968)	0.8262
Byatt-Smith (1970)	0.83
Strelkoff (1970)	0.85
Fenton (1972)	0.85
Longuet-Higgins & Fenton (1974)	0.827
Williams (1981)	0.833197
Witting (1981)	0.8332
Hunter & Vanden-Broeck (1983)	0.83322

TABLE 1

recently Hunter & Vander-Broeck (1983) have calculated the maximum wave to be 0.83322, based on the earlier work of Lenau (1966). They have also recalculated the results of Yamada (1957) and Byatt-Smith & Longuet-Higgins (1976).

In this paper we study the properties of solitary waves by means of a perturbation scheme. Our approach resembles that of Longuet-Higgins & Fenton, but we derive our series coefficients by a different method. Based on the results of earlier investigations, we assume that the fluid velocity on the channel bed can be expressed in a power series of hyperbolic secants, which enables us to formulate the problem as an infinite system of nonlinear algebraic equations. This derivation is given in §2, where we also discuss the partial numerical solution of these equations. This partial solution allows us to find the first 17 terms in a series expansion for the wave profile. The computational results are presented in §4. In §5 we compare this method with other recent methods.

2. Formulation of the problem

We consider the two-dimensional irrotational flow of an inviscid fluid of uniform density in an open channel with a horizontal bottom. The x-axis is taken to lie along the channel bed and the y-axis is taken vertically upward. The shape of the free surface of the fluid is given by

$$y = h(t,x) = h_0[1+\zeta(t,x)], \tag{2.1}$$

where h_0 denotes the depth of the fluid in the undisturbed state. Since the flow is incompressible and irrotational there exists a harmonic velocity potential $\phi(t,x,y)$. Expanding ϕ about $y=0$ and using the facts that ϕ is harmonic and that $\partial\phi/\partial y = 0$ on $y=0$, we obtain

$$\phi(t,x,y) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left[\frac{\partial^{2n}}{\partial x^{2n}}\Phi(t,x)\right]y^{2n}, \tag{2.2}$$

where $\Phi(t,x) = \phi(t,x,0)$. Since we are interested specifically in stationary-wave solutions, we assume that $\Phi(t,x) = \Phi(\xi)$ and $\zeta(t,x) = \zeta(\xi)$, with $\xi = (\kappa/h_0)(x-ct)$.

($1/\kappa$ measures the non-dimensional width of the wave.) We also introduce the notation

$$\mathrm{D} = \frac{\mathrm{d}}{\mathrm{d}\xi}, \quad w(\xi) = \frac{1}{c}\frac{\partial \Phi}{\partial x}, \tag{2.3), (2.4}$$

$$u_{\mathrm{s}}(\xi) = \frac{1}{c}\frac{\partial \phi}{\partial x}\bigg|_{y=h(\xi)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(\mathrm{D}^{2n}w)\,[\kappa(1+\zeta)]^{2n}, \tag{2.5}$$

$$v_{\mathrm{s}}(\xi) = \frac{1}{c}\frac{\partial \phi}{\partial y}\bigg|_{y=h(\xi)} = -\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(\mathrm{D}^{2n+1}w)\,[\kappa(1+\zeta)]^{2n+1}. \tag{2.6}$$

In this notation the kinematic free-surface boundary condition is

$$\kappa\zeta = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(\mathrm{D}^{2n}w)\,[\kappa(1+\zeta)]^{2n+1} \tag{2.7}$$

and the dynamic free-surface condition is

$$-u_{\mathrm{s}} + \tfrac{1}{2}(u_{\mathrm{s}}^2 + v_{\mathrm{s}}^2) + \frac{c_0^2}{c^2}\zeta = 0, \tag{2.8}$$

where $c_0^2 = gh_0$ and g is the acceleration due to gravity.

Fenton (1972) noticed that, based on the solutions of Boussinesq, Laitone (1960) and Grimshaw (1971), it seemed reasonable to express ζ as a series in powers of $\operatorname{sech}^2 \tfrac{1}{2}\xi$. He also noted that differentiating such a series an even number of times would yield a series of the same form. This suggests the possibility of expanding the velocity w in a series

$$w = \sum_{m=1}^{\infty} a_m S^m, \tag{2.9}$$

with $S = \operatorname{sech}^2 \tfrac{1}{2}\xi$. This expansion may be shown to be complete in the L^2 sense (see Appendix B). Substituting (2.9) into (2.7) and summing over n, we obtain

$$\kappa\zeta = \sum_{m=1}^{\infty} \alpha_m F_m(S) \sin m\kappa(1+\zeta), \tag{2.10}$$

where

$$\alpha_m = (-1)^m \sum_{l=1}^{m} (-4)^l \binom{m+l-1}{m-l} a_l \tag{2.11}$$

and

$$mF_m(S) = (\tfrac{1}{4}S)^m F(m, m+\tfrac{1}{2}; 2m+1; S) = \exp(-m|\xi|). \tag{2.12}†$$

We can invert (2.11) and obtain

$$a_m = 4^{-m} \sum_{l=1}^{m} \frac{l}{m}\binom{2m}{m-l}\alpha_l.$$

Using this and (2.12), we can rewrite a power series in S as a power series in $\exp(-|\xi|)$, e.g.

$$w = \sum_{n=1}^{\infty} a_n S^n = \sum_{n=1}^{\infty} \alpha_n \mathrm{e}^{-n|\xi|}. \tag{2.13}$$

In §5 we shall use these results to discuss the discrepancy in the literature of the highest solitary wave between the works based on a series solely in S and those that include explicitly some terms of the form $\mathrm{e}^{-n|\xi|}$ to model the discontinuity in surface slope at the crest.

† $F(m, m+\tfrac{1}{2}; 2m+1; S)$ is a hypergeometric function.

We go on now to find expressions similar to (2.10) for u_s and v_s from (2.5) and (2.6), i.e.

$$u_s = \sum_{m=1}^{\infty} m\alpha_m F_m(S) \cos m\kappa(1+\zeta) \tag{2.14}$$

and

$$v_s = \sum_{m=1}^{\infty} \alpha_m (\mathrm{D}F_m) \sin m\kappa(1+\zeta). \tag{2.15}$$

We are now in a position to describe our method of solution. We first use (2.10), (2.14) and (2.15) to express ζ, u_s and v_s as series in powers of S, the coefficients of which are functions of the unknowns α_m. Next we substitute these expressions into the dynamic boundary condition (2.8), collect powers of S and set the coefficient of each power of S to zero, thereby producing a system of equations in the α_m. Solving these equations completes the solution.

Using the Lagrange expansion theorem to invert (2.10), we find that

$$\kappa\zeta = (\tfrac{1}{4}S) \sum_{n=0}^{\infty} (\tfrac{1}{4}S)^n \left[\sum_{m=1}^{n+1} Z_{n,m} \sin m\kappa\right], \tag{2.16}$$

where $Z_{n,m}$ is an algebraic combination of $\alpha_1, \ldots, \alpha_{n+1}$. Similarly, it follows from (2.14) and (2.15) that

$$u_s = (\tfrac{1}{4}S) \sum_{n=0}^{\infty} (\tfrac{1}{4}S)^n \left[\sum_{m=0}^{n+1} U_{n,m} \cos m\kappa\right] \tag{2.17}$$

and

$$v_s = -(\tfrac{1}{4}S)' \sum_{n=0}^{\infty} (\tfrac{1}{4}S)^n \left[\sum_{m=1}^{n+1} V_{n,m} \sin m\kappa\right]. \tag{2.18}$$

To derive the equations that will enable us to solve for the α_m, we replace ζ, u_s and v_s in (2.8) by (2.16), (2.17) and (2.18). Setting the coefficient of $\frac{1}{4}S$ equation zero yields

$$0 = -\alpha_1 \cos\kappa + \frac{c_0^2}{\kappa c^2}\alpha_1 \sin\kappa, \quad \text{or} \quad \frac{c^2}{c_0^2} = \frac{\tan\kappa}{\kappa}. \tag{2.19), (2.20}$$

($\alpha_1 = 0$ leads to a trivial solution.) Equating the higher powers of $\frac{1}{4}S$ to zero and using (2.20), we find that

$$0 = \sum_{m=1}^{n+1} E_{n,m} \sin m\kappa \quad \text{for } n \geqslant 2, \tag{2.21}$$

where $E_{n,m}$ is an algebraic combination of $\alpha_1, \ldots, \alpha_n$. We shall refer to (2.21) as the α-equations. It can be shown that for odd n (even n) the sum of all terms in (2.21) with odd m (even m) equals a linear combination of α-equations $n-1$, $n-3$, ..., and is therefore zero. This being the case, we can rewrite the α-equations in such a way that κ appears only in the form of powers of $\sin^2\kappa$. For example, for $n = 2$, 3 and 4 we have

$$\tfrac{3}{2}\alpha_1^2 + (\sin^2\kappa)\,\alpha_2 = 0, \quad 3\alpha_1(3\alpha_2 - 2\alpha_1^2) + 8\,(\sin^2\kappa)\,\alpha_3 = 0 \tag{2.22), (2.23}$$

and

$$\tfrac{3}{2}\alpha_2^2 + 3\alpha_1\alpha_3 + \tfrac{3}{4}\alpha_1^2\alpha_2 - 12\alpha_1^4 + (\sin^2\kappa)\,[(5-6\sin^2\kappa)\,\alpha_4 - \tfrac{5}{4}\alpha_1^4] = 0. \tag{2.24}$$

The converted α-equations form an infinite system of algebraic equations for the α_m, the coefficients being polynomials in $\sigma = \sin^2\kappa$. This system may be regarded as a set of recurrence relations. A straightforward method of solution would be to use the nth equation to solve for α_n in terms of the unspecified quantity α_1. We note, however, that in the nth equation the coefficient of α_n (namely, $n^{-1}\cot\kappa \sin n\kappa - \cos n\kappa$) has a double zero at $\kappa = 0$ and $[\frac{1}{2}n-1]$ other zeros distributed more or less evenly over each interval $\frac{1}{2}m\pi \leqslant \kappa \leqslant \frac{1}{2}(m+1)\pi$, $m = 0, \pm 1, \pm 2, \ldots$. For example, consider the converted equation for $n = 4$, (2.24). The coefficient of α_4 has a zero at $\kappa = \sin^{-1}\sqrt{\frac{5}{6}}$. Furthermore, the remaining terms do not sum to zero when $\kappa = \sin^{-1}\sqrt{\frac{5}{6}}$, so it is impossible to find a 'straightforward' solution at this value of κ. As $n \to \infty$ the zeros of the coefficient of α_n in equation n cover the interval $-\infty < \kappa < \infty$ more and more densely, and the set of values of κ at which the 'straightforward' method fails is actually dense in $(-\infty, \infty)$. Thus we are forced to go back to the perturbation method of solution based on the shallow-water approximation originated by Boussinesq and Korteweg & deVries and further developed by Laitone (1960), Grimshaw (1971) and Fenton (1972). We will attempt to find a Taylor-series expansion for each of the α_m, and α_1 will no longer be arbitrary. In Appendix A we show how this method is connected to the recurrence relation method in a simple problem for which the explicit solution can be found.

Following Fenton, we express each coefficient a_m (from (2.9)) as a series in powers of σ:

$$a_m = \sum_{n=m}^{\infty} a_{m,n}\sigma^n \tag{2.25}$$

(note that $a_m = O(\sigma^m)$). This implies that

$$\alpha_m = \sum_{n=1}^{\infty} \alpha_{m,n}\sigma^n, \tag{2.26}$$

where, however, $\alpha_{m,m-l}$ $(1 \leqslant l \leqslant m-1)$ may be determined from $\alpha_{1,m-l}, \ldots, \alpha_{m-l,m-l}$ by means of (2.11) and (2.25). We use the first N α-equations to obtain partial series expansions for the unknowns $\alpha_1, \ldots, \alpha_{N+1}$:

$$\alpha_i \sim \alpha_{i,1}\sigma + \ldots + \alpha_{i,N}\sigma^N. \tag{2.27}$$

This is accomplished by substituting (2.26) into the (converted) α-equations and collecting powers of σ. By setting the coefficients of σ^{N+1} from the first N equations equal to zero, we generate a system of N linear algebraic equations for the N unknown quantities $\alpha_{1,N}, \ldots, \alpha_{N,N}$.

We now have, at least in principle, a method for determining $\alpha_{m,n}$, $1 \leqslant m, n < \infty$. The method proceeds in stages, the nth stage consisting of the following steps: (1) generate $Z_{n,j}$, $U_{n,j}$ and $V_{n,j}$, $0 \leqslant j \leqslant n+1$, in (2.16)–(2.18); (2) generate the α-equation coefficients $E_{n+1,j}$, $1 \leqslant j \leqslant n+2$; and (3) generate and solve the linear system for $\alpha_{1,n}, \ldots, \alpha_{n,n}$. Although computation by hand becomes impractical for n greater than four or five, this procedure is sufficiently mechanical for implementation on a computer. In this way we were able to carry out the calculations for $n \leqslant 17$. The results of the calculations are presented in §4.

3. Round-off error

The only source or error in our calculations thus far is accumulated round-off error. We use three methods for estimating round-off at each order of our procedure. The first method is based upon the fact that the sum

$$E_{n,n}\sin n\kappa + E_{n,n-2}\sin(n-2)\kappa + \ldots + E_{n,t}\sin t\kappa \tag{3.1}$$

($t = 1$ for odd n, $t = 2$ for even n) from the nth α-equation equals a linear combination of the right-hand sides of α-equations $n-1, n-3, \ldots, t+1$. We subtract the

K \ M	1	2	3	4
1	0.33333333	0.16666667	0.11018519	0.072641675
2	0	0.083333332	0.055092601	0.043507215
3	0	0	0.046759250	0.047392988
4	0	0	0	0.026800920

K \ M	5	6	7	8
1	0.042451952	0.015943894	−0.0085614296	−0.032035128
2	0.036361955	0.031004546	0.026444495	0.022195438
3	0.047218344	0.045727388	0.042584726	0.037680372
4	0.033407458	0.037430332	0.040145228	0.041726872
5	0.017476811	0.026543319	0.033510491	0.039074886
6	0	0.011972623	0.021386065	0.030256270
7	0	0	0.0085084173	0.027202015
8	0	0	0	0.0063118728

K \ M	9	10	11
1	−0.055153658	−0.078468449	−0.10249052
2	0.017971190	0.013577586	0.0088654933
3	0.031016670	0.022653500	0.012680075
4	0.042160786	0.041377625	0.039283857
5	0.043068759	0.045062503	0.044463739
6	0.039301862	0.049367581	0.062392013
7	0.025566166	0.031326842	0.026892083
8	0.014994098	0.028340832	0.059776797
9	0.0045866638	0.010005264	0.0014379156
10	0	0.0040313058	0.019676312
11	0	0	0.00086393297

K \ M	12	13	14
1	−0.12774163	−0.15479179	−0.18429265
2	0.0037068090	−0.0020197882	−0.0084386890
3	0.0011981582	−0.011689462	−0.025888042
4	0.035771128	0.030719535	0.023998437
5	0.040491666	0.032079595	0.017696353
6	0.082986836	0.12170174	0.20199787
7	−0.011133411	−0.15292174	−0.60501210
8	0.17055174	0.61916484	2.4427726
9	−0.12030501	−0.94865989	−5.7439439
10	0.14140336	1.2168236	9.8041296
11	−0.046338463	−0.87568627	−10.922887
12	0.012928512	0.39238498	7.8421140
13	0	−0.070328864	−3.2040519
14	0	0	0.57606575

K \ M	15	16	17
1	−0.21701156	−0.25386774	−0.29598301
2	−0.015685214	−0.023914058	−0.033303809
3	−0.041318858	−0.057927173	−0.075690417
4	0.015465755	0.0049658524	−0.0076736184
5	0.0049502296	−0.039393295	−0.091177063
6	0.37518132	0.75234009	1.5725370

$K \backslash M$	15	16	17
7	−1.9661929	−5.9518985	−17.421640
8	9.6594701	37.449813	142.11681
9	−31.392266	−161.73222	−800.27067
10	71.950849	489.77232	3150.2774
11	−112.81499	−1034.1499	−8717.7604
12	119.06500	1512.1546	16990.411
13	−80.366460	−1496.2086	−23128.098
14	31.351102	954.2586	21481.010
15	−5.3613204	−353.83078	−12954.800
16	0	57.854339	4567.9727
17	0	0	−714.22617

TABLE 2. Coefficients $P_{k,m}$ in the expansion
$\zeta = \sum_{k=1}^{\infty} \sigma^k \left[\sum_{m=1}^{k} P_{k,m} S^m \right]$ for $1 \leqslant m \leqslant k,\ 1 \leqslant k \leqslant 17$

appropriate combination from (3.1) and look at the coefficient of each term; the largest deviation from zero is an estimate of the error involved in finding $\alpha_{m,n}$, $1 \leqslant m \leqslant n$. As a second method of estimating error, we apply the dynamic boundary condition (2.8) at the wave crest, expanding each term as a series in powers of σ and collecting powers of σ. The deviation of the coefficient of σ^n from zero is a measure of the nth-order error. As a final test, we employ an identity first proved by Starr (1947) and also used by Longuet-Higgins & Fenton to approximate round-off:

$$0 = 3\tilde{V} - \left(\frac{c^2}{c_0^2} - 1\right)\tilde{M}, \tag{3.2}$$

where $\tilde{V}$ is potential energy and $\tilde{M}$ is mass (see §4 for precise definitions). We expand each term on the right-hand side in powers of σ and collect powers of σ. The size of the coefficient of σ^n is a measure of round-off at the nth order of the procedure. Based on these estimates, it appears that our computations are correct to more than 10 significant digits up to order 14 and to 6 digits thereafter.

4. Computational results

Using (2.7) and the values of $\alpha_{i,j}$, $1 \leqslant i, j \leqslant 17$, we can compute the coefficients in the series

$$\zeta = \sum_{k=1}^{\infty} \sigma^k \left[\sum_{m=1}^{k} P_{k,m} S^m \right] \tag{4.1}$$

for $1 \leqslant k \leqslant 17$. The coefficients $P_{k,m}$ are listed in table 2. From (4.1) we can derive expressions for the amplitude ϵ, mass $\tilde{M}$ and potential energy $\tilde{V}$ of the solitary wave:

$$\epsilon = \zeta|_{\xi=0} = \sum_{k=1}^{\infty} \epsilon_k \sigma^k, \tag{4.2}$$

$$\tilde{M} = \frac{M}{h_0^2} = \sigma^{-\frac{1}{2}} \sum_{k=1}^{\infty} \mu_k \sigma^k, \tag{4.3}$$

$$\tilde{V} = \frac{V}{c_0^2 h_0^2} = \sigma^{\frac{1}{2}} \sum_{k=1}^{\infty} \nu_k \sigma^k, \tag{4.4}$$

n	ϵ_n	ϵ'_n	μ_n	μ'_n	ν_n	ν'_n
1	0.33333333	0.333333	1.33333333	1.333333	0.14814815	0.148148
2	0.25000000	0.250000	0.66666667	0.666667	0.18271605	0.182716
3	0.21203704	0.212037	0.47629630	0.476296	0.19707231	0.197072
4	0.19034281	0.190343	0.36779189	0.367792	0.20288105	0.202881
5	0.17691656	0.176917	0.28970821	0.289708	0.20368768	0.203688
6	0.16862220	0.168622	0.22606847	0.226069	0.20095335	0.200953
7	0.16401820	0.164018	0.16985500	0.169855	0.19535807	0.195358
8	0.16241302	0.162413	0.11723581	0.117236	0.18721160	0.187212
9	0.16351324	0.163513	0.06572742	0.065728	0.17661233	0.176612
10	0.16727555	0.167275	0.01345792	0.013458	0.16351567	0.163516
11	0.17384262	0.173843	−0.04118564	−0.041185	0.14776331	0.147764
12	0.18351974	0.183520	−0.09975613	−0.099756	0.12909281	0.129093
13	0.19677393	0.196774	−0.16387960	−0.163879	0.10713548	0.107136
14	0.21425084	0.21425	−0.23536227	−0.2354	0.08140565	0.0814
15	0.23679823	0.237	−0.31629150	−0.32	0.05128254	0.05
16	0.265532		−0.409139		0.015985	
17	0.301774		−0.516891		−0.025463	

TABLE 3. Coefficients in the expansions of ϵ, $\tilde{M}$ and $\tilde{V}$ in powers of σ (the prime denotes the coefficients calculated by Longuet-Higgins & Fenton 1974)

where M and V are given by

$$M = \int_{-\infty}^{\infty} h_0\, \zeta(t,x)\,\mathrm{d}x = \frac{h_0^2}{\kappa}\int_{-\infty}^{\infty} \zeta(\xi)\,\mathrm{d}\xi \tag{4.5}$$

and

$$V = \int_{-\infty}^{\infty}\int_{h_0}^{h_0(1+\zeta)} gy\,\mathrm{d}y\,\mathrm{d}x - Mg\,h_0 = \frac{1}{2}\,\frac{c_0^2 h_0^2}{\kappa}\int_{-\infty}^{\infty} [\zeta(\xi)]^2\,\mathrm{d}\xi. \tag{4.6}$$

The coefficients ϵ_k, μ_k and ν_k for $k \leqslant 17$ are listed in table 3. For purposes of comparison, we have included the corresponding coefficients as calculated from the data given by Longuet-Higgins & Fenton (1974).

For small- to medium-amplitude waves, these series expressions provide perfectly adequate descriptions of the wave-profile and parameter values. Series (4.1) and (4.2) give two-digit accuracy even at $\epsilon \approx 0.52$ (as determined by observed convergence of the partial sums). For large-amplitude waves, however, a series in powers of $\sin^2\kappa$ cannot possibly be accurate, since more than one wave corresponds to a single value of κ for κ larger than some critical value. If we follow Longuet-Higgins & Fenton, recasting our series in terms of the parameter $\omega = 1-(c^2/c_0^2)\,[1-u_s(0)]^2$ and using diagonal Padé approximants to accelerate convergence, we find that it is possible to estimate the values of ϵ, $\tilde{M}$, $\tilde{V}$ and c^2/c_0^2 for waves of amplitudes up to and including the maximum. The observed convergence of the Padé approximants is not completely convincing, however, and we feel that more than 17 terms are required if this method is to be used with confidence. Our computed values agree with those given by Longuet-Higgins & Fenton (1974), but differ from the more recent calculations of Williams (1981), Witting (1981) and Hunter & Vanden-Broeck (1983). (See the discussion below.)

5. Summary and discussion

We compute the first 17 terms in a series expressing the free-surface elevation ζ in powers of $\operatorname{sech}^2 \frac{1}{2}k(x-ct)$. For small- to medium-amplitude waves this series provides accurate values of wave parameters and profiles. By using Padé approximants to accelerate convergence, we can calculate parameter values for waves of all amplitudes. Our calculations agree with those of Longuet-Higgins & Fenton (1974), but differ from the data given by Williams (1981), Witting (1981) and Hunter & Vanden-Broeck (1983). These authors include terms similar to $e^{-m|\xi|}$ to account for the sharp crest of the highest wave. Witting (1975) suggests a possible explanation for this discrepancy: expansions such as (4.1) that use only one value of σ (or κ) are incomplete and are only valid asymptotically as ϵ approaches zero (cf. Miles 1980; Schwarz & Fenton 1982). He reaches this conclusion after demonstrating the inadequacy of a solution procedure based on the expansion

$$x+\mathrm{i}y = \phi+\mathrm{i}\psi+\sum_{n=1}^{\infty} a_n \mathrm{e}^{n\kappa(\phi+\mathrm{i}\psi)}. \tag{5.1}$$

The a_n must satisfy a system of equations similar to (2.21). Witting regards the system as a set of recurrence relations and solves for a_n $(n \geqslant 2)$ in terms of the arbitrarily chosen a_1. This method is subject to the objection outlined in §2, however. Furthermore, it can be shown that the set of functions $\{\operatorname{sech}^{2n}\frac{1}{2}x\}_{n=1}^{\infty}$ spans $L^2(0,\infty)$ (see Appendix B). Thus it would appear that we can expect (4.1) to converge in the mean. Our data suggest that we may even have pointwise convergence, at least for low- to medium-amplitude waves. Also, since the series for $e^{-n|\xi|}$ in powers of S converges pointwise on $-\infty<\xi<\infty$ (Appendix B), (4.1) will converge pointwise even for the highest wave if it proves possible to describe that wave in terms of the functions $\{e^{-n|\xi|}\}_{n=1}^{\infty}$.

From (2.12) we see that an exponential function that has a discontinuity at $\xi = 0$ can be expressed as a hypergeometric function that has a branch-point singularity at $S = 1$ (or $\xi = 0$), i.e.

$$F(m, m+\tfrac{1}{2}; 2m+1; S) \approx 4^m[1-2m(1-S)^{\frac{1}{2}}] \quad \text{for} \quad S \approx 1. \tag{5.2}$$

The hypergeometric series of F in S converges very slowly near $S = 1$. One may also raise the question of how well a rational-function (Padé) approximation can represent a branch-point singularity. This may cast some doubt on the series solution in S as given by the first part of (2.13) in the case of the maximum solitary wave, which contains a discontinuity of slope at the crest. On the other hand to include any term in the second part of (2.13) will introduce a discontinuity of slope, which is not acceptable for any waves other than the maximum one. We believe that one way to resolve this dilemma is to carry the expansion of (2.13) either in a_n or α_n to sufficiently high order that the nature of the branch-point singularity can be extracted from the series. The two distinct expansions will then become one.

This research was sponsored by the Office of Naval Research, U. S. Navy, and the National Science Foundation (CME-801-2192, Fluid Dynamics Program).

Appendix A. Example of a recurrence relation containing a small parameter

Consider the recurrence relation

$$n\epsilon A_n = A_{n-1} - \frac{1}{(n-1)!} \quad (n \geqslant 2). \tag{A 1}$$

This system cannot be solved by the 'straightforward method' referred to in §2 when $\epsilon = 0$. For $\epsilon \neq 0$ the system can be solved exactly i.e.

$$\begin{aligned} A_n &= \frac{1}{n!}\frac{1}{\epsilon^{n-1}}[A_1 - 1 - \epsilon - \ldots - \epsilon^{n-2}] \\ &= \frac{1}{n!}\frac{1}{\epsilon^{n-1}}\left[A_1 - \frac{1-\epsilon^{n-1}}{1-\epsilon}\right]. \end{aligned} \tag{A 2}$$

If A_n in (A 2) is to remain finite as $\epsilon \to 0$, we must have $A_1 = 1/(1-\epsilon) + O(\epsilon^{n-1})$. Taking $A_1 = 1/(1-\epsilon)$ in (A 2), we find that

$$A_n = \frac{1}{n!}\frac{1}{1-\epsilon}. \tag{A 3}$$

On the other hand, we can ignore solution (A 2) and attempt to solve (A 1) by expressing each A_n as a series in powers of ϵ:

$$A_n = \sum_{m=0}^{\infty} A_{n,m}\,\epsilon^m \quad (n \geqslant 1). \tag{A 4}$$

Substituting (A 4) into (A 1) and setting the coefficients of each power of ϵ equal to zero, we find that $A_{n,m} = 1/n!$. Therefore

$$A_n = \frac{1}{n!}(1 + \epsilon + \epsilon^2 + \ldots) = \frac{1}{n!}\frac{1}{1-\epsilon}. \tag{A 5}$$

This is the same as (A 3), which we found by working with the exact solution (A 2). We cannot find the explicit solution of the more complicated system (2.21), so we only use the series-expansion method (§2).

Appendix B. Proof that $\{\operatorname{sech}^{2n}\tfrac{1}{2}x\}_{n=1}^{\infty}$ spans $L^2(0,\infty)$

LEMMA 1. *For all f in $L^2(0,\infty)$ and for all $\epsilon > 0$, there exists a polynomial*

$$p(\mathrm{e}^{-2x}) = \sum_{m=0}^{n} A_m\,\mathrm{e}^{-2mx}$$

such that

$$\|f - \mathrm{e}^{-x}p(\mathrm{e}^{-2x})\| < \epsilon.$$

($\| \;\|$ denotes the L^2 norm.)

LEMMA 2.

$$\mathrm{e}^{-nx} = \sum_{m=n}^{\infty}\left(\frac{1}{4}\right)^m \frac{n}{m}\binom{2m}{m-n}\operatorname{sech}^{2m}\tfrac{1}{2}x \quad \text{in } L^2(0,\infty).$$

PROPOSITION.

$$V\{\operatorname{sech}^{2n}\tfrac{1}{2}x\}_{n=1}^{\infty} = L^2(0,\infty).$$

Proof of proposition: Let $f \in L^2(0,\infty)$ and $\epsilon > 0$ be given. We need to show that there exists a polynomial $q(S)$ (where $S \equiv \operatorname{sech}^2\tfrac{1}{2}x$) such that $\|f - q(S)\| < \epsilon$. By lemma 1, $\exists p(\mathrm{e}^{-2x}) = \sum_{m=0}^{n} A_m\,\mathrm{e}^{-2mx}$ such that $\|f - \mathrm{e}^{-x}p(\mathrm{e}^{-2x})\| < \tfrac{1}{2}\epsilon$. By lemma 2, $\exists p_m(S)$ such that

$$\|\mathrm{e}^{-(2m+1)x} - p_m(S)\| < \frac{\epsilon}{2(n+1)\,|A_m|} \quad (m = 0, \ldots, n)$$

(assuming $A_m \neq 0$). Therefore

$$\begin{aligned}\left\|f-\sum_{m=0}^{n} A_m p_m(S)\right\| &\leqslant \|f-\mathrm{e}^{-x} p(\mathrm{e}^{-2x})\| + \left\|\mathrm{e}^{-x} p(\mathrm{e}^{-2x}) - \sum_{m=0}^{n} A_m p_m(S)\right\| \\ &< \tfrac{1}{2}\epsilon + \sum_{m=0}^{n} \|A_m \mathrm{e}^{(2m+1)x} - A_m p_m(S)\| \\ &\leqslant \tfrac{1}{2}\epsilon + \sum_{m=0}^{n} |A_m| \frac{\epsilon}{2(n+1)|A_m|} \\ &= \tfrac{1}{2}\epsilon + \tfrac{1}{2}\epsilon = \epsilon.\end{aligned}$$

$\Sigma_{m=0}^{n} A_m p_m(S)$ is the desired polynomial $q(S)$. Q.E.D.

Proof of lemma 1: Let $f \in L^2(0,\infty)$ and $\epsilon > 0$ be given. Let $g(x) \equiv \mathrm{e}^x f(x)$. Then

$$\int_0^\infty \mathrm{e}^{-2x} g^2 \,\mathrm{d}x = \int_0^\infty f^2 \,\mathrm{d}x \equiv M < \infty.$$

Define $$g_n(x) = g(x)\,\chi_{[1/n,\,n]}, \quad n = 1,2,3,\ldots.$$

Then $$\int_0^\infty \mathrm{e}^{-2x} g_n^2(x)\,\mathrm{d}x \leqslant \int_0^\infty \mathrm{e}^{-2x} g^2(x) = M.$$

Now let $y = \mathrm{e}^{-2x}$ and $h_n(y) = g(-\tfrac{1}{2}\ln y)\,\chi_{[\mathrm{e}^{-2n},\,\mathrm{e}^{-2/n}]}, \quad 0 < y \leqslant 1.$

Then $$\begin{aligned}M \geqslant \int_0^\infty \mathrm{e}^{-2x} g_n^2 \,\mathrm{d}x &= \int_{1/n}^{n} \mathrm{e}^{-2x} g^2 \,\mathrm{d}x \\ &= \frac{1}{2}\int_{\mathrm{e}^{-2n}}^{\mathrm{e}^{-2/n}} g^2(-\tfrac{1}{2}\ln y)\,\mathrm{d}y \\ &= \frac{1}{2}\int_0^1 h_n^2(y)\,\mathrm{d}y.\end{aligned}$$

Thus $$\int_0^1 h_n^2 \,\mathrm{d}y \leqslant 2M \quad \text{and} \quad h_n^2 \uparrow g^2(-\tfrac{1}{2}\ln y)$$

almost everywhere on (0, 1). Therefore, by the Monotone Convergence Theorem,

$$\int_0^1 g^2(-\tfrac{1}{2}\ln y) = \lim_{n\to\infty} \int_0^1 h_n^2 \,\mathrm{d}y \leqslant 2M,$$

i.e. $g(-\tfrac{1}{2}\ln y) \in L^2(0,1)$. Since $L^2(0,1) = V\{y^n\}_{n=1}^{\infty}$, there exists a polynomial $p(y)$ such that

$$\int_0^1 [g(-\tfrac{1}{2}\ln y) - p(y)]^2 \,\mathrm{d}y < 2\epsilon^2.$$

Replacing y by e^{-2x}, we find that

$$2\int_0^\infty [g(x) - p(\mathrm{e}^{-2x})]^2 \mathrm{e}^{-2x} \,\mathrm{d}x < 2\epsilon^2, \quad \text{or} \quad \int_0^\infty [\mathrm{e}^{-x} g(x) - \mathrm{e}^{-x} p(\mathrm{e}^{-2x})]^2 \,\mathrm{d}x < \epsilon^2.$$ Q.E.D.

Proof of lemma 2: Using (2.12), we obtain the formula

$$\mathrm{e}^{-nx} = \sum_{m=n}^{\infty} \frac{n}{m}\left(\frac{1}{4}\right)^m \binom{2m}{m-n} S^m \equiv \sum_{m=n}^{\infty} B_{n,m} S^m.$$

This series converges pointwise for all S in the range $0 \leqslant S \leqslant 1$. Thus in order to prove L^2 convergence it suffices to show that the norms of the partial sums converge to the norm of e^{-nx}. Note that

$$\left\| \sum_{m=n}^{n+N} B_{n,m} S^m \right\|^2 = \int_0^\infty \left[\sum_{m=n}^{n+N} B_{n,m} S^m \right]^2 dx$$

$$= \int_0^\infty \sum_{m=2n}^{2(n+N)} C_{n,N,m} S^m \, dx.$$

Since

$$C_{n,N,m} \begin{cases} = B_{2n,m} & \text{for} \quad 2n \leqslant m \leqslant 2n+N, \\ \leqslant B_{2n,m} & \text{for} \quad 2n+N < m \leqslant 2(n+N), \end{cases}$$

it follows that

$$\sum_{m=2n}^{2(n+N)} C_{n,N,m} S^m \uparrow e^{-2nx} \quad (\text{as } N \to \infty) \quad \text{pointwise on} \quad 0 \leqslant x < \infty.$$

Hence by the Monotone Convergence Theorem

$$\int_0^\infty \sum_{m=2n}^{2(n+N)} C_{n,N,m} S^m \to \int_0^\infty e^{-2nx} \, dx,$$

i.e.

$$\left\| \sum_{m=n}^{n+N} B_{n,m} S^m \right\|^2 \to \| e^{-nx} \|^2. \qquad \text{Q.E.D.}$$

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